International Journal of Civil Engineering and Environmental Science. Volume 2, Issue 1, (2014) 12-22
http://www.cayley-nielson.com/ijcees/

Governmental Guarantee Option Value of Utility Indifference Pricing

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Abstract

Government guarantee is one of important measures to attract investment in infrastructure for developing countries. Most of existing researches are based on the hypothesis of complete market, but this paper has broken traditional risk neutral hypothesis. Considering investors' risk attitude in incomplete market, the utility indifference pricing has been introduced to the option value of minimum traffic guarantee in highway. This paper constructs the European option pricing model in government guarantee and offers partial differential equation for utility indifference pricing point. Meanwhile, through the analysis of partial differential equation for utility indifference pricing under CARA utility functions, the pricing method given by this paper has been proved consistent with traditional B-S equation pricing method under the condition of investors being risk neutral. In the end, through the numerical solution to partial differential equation, an interesting relation between option price of minimum traffic guarantee and investors' risk attitude has been revealed.

Keywords: Government Guarantee, Utility Indifference Pricing, CARA Utility, Risk Aversion.

1. Introduction

In Traffic transportation "Twelfth Five-Year" development planning, the Ministry of Transport has pointed out that to the year of 2015, the total length of highway will grow from 74 thousand km in 2010 to 108 thousand km, government has to broaden financing channel to alleviate the pressure of capital demand. Take Shaanxi province as an example, Transport Department is planning on the largest highway BOT project in history, with a total length of 1530 km and investment of 159 billion RMB, which will be a significant investment opportunity for private investors. However highway investment is characterized by large investment and long cost recovery cycle, Klein (1997) [1]researched on business risk and policy risk in infrastructure projects and said that for most of the developing countries, without government guarantee there is no way to raise large medium and long-term investment for infrastructure construction; also according to Liantoet al. (1997)[2], the government guarantee can reduce or eliminate those risk factors which prevent the private sectors to participate in public infrastructure projects. If the government wants to attract private investment in highway construction, it needs to provide certain conditional government guarantee, but the government guarantee is not optionally provided free resource, when some particular event occurs in the future, this contingent responsibility will appear, demanding government financial payment [3]. Therefore how to calculate the option value of government guarantee according to the market situation and relevant risk factors, so as to reasonably determine the guarantee fee or investment plan has become an urgent problem to be solved.

ISSN 2333-5416 /
March, 2014

In project investment analysis, NPV method is the most widely used project assessment method, but it has a fatal defect which is the ignorance of magnitude and degree of asset risk. For example, in the guarantee contract of minimum highway traffic flow, it' s almost impossible to predict an accurate traffic flow. Myers (1977) [4]proposed the concept of real options for the first time, and introduced the idea of option into investment analysis. Wibowo (2004) [5]analyzed several guarantee forms in BOT infrastructure projects: minimum income guarantee, minimum traffic guarantee, charge adjustment guarantee and debt repayment guarantee, through which he thought guarantee could be seen as a put option and had value calculation. Huang et al. (2006) [6]analyzed the value of abandonment option and government-offered minimum income guarantee in BOT project and the influence of interaction on each value, he thought the minimum income guarantee that government provided to the project could be seen as European put option, and could be calculated with B-S option pricing method. Brandao and Saraiva (2008) [7]constructed a minimum traffic guarantee real options model and applied this model to the projected 1000 mile long BR－163 toll road. Galera and Solino (2010) [8]also used the real options theory to work on the value of highway minimum traffic guarantee. Zhang et al. (2011) [9]developed a value model of minimum revenue guarantee with multiple-exercise real options under the impact of the emergency incident.

The traditional real options pricing method is based on the assumption of complete market, that is, all risk assets can be completely copied, and then option value will be in found under equivalent martingale measure or no-arbitrage principle [8, 10, 11]. However, implied assets in highway guarantee options can't be traded, so the value of the option can't be completely copied, which means the market is incomplete. When it's an incomplete market, equivalent martingale measure is no longer exclusive, and so is the option value. To this situation, Hodges and Neuberger (1989) [12]proposed the idea of utility indifference pricing. This pricing method, based on utility view, aiming at unreachable (unduplicated) contingent claims, has taken investors' subjective risk awareness under consideration, which can solve the option pricing problem in incomplete market situation.

There are lots researches studying option pricing with utility indifference pricing theory, which can be generally divided in two kinds, one is pricing research under the principle of utility maximization of ultimate wealth expectation[13-16], but this research has neglected the truth that consumption would impact on the investors' decision. The other one is option pricing on the basis of expected consumption utility maximization, but under an unlimited duration condition[17], which is apparently different from reality. Among documents comprehensively considering both of the expected consumption utility and ultimate wealth utility maximization, most of the researches focus on optimal investment and consumption strategies, rarely is on option pricing.

The key to pricing with utility indifference theory is solving maximum utility functions; there are mainly two ways for this research. The first one, also mostly used one, is conjecture-verification method, which first began in 1969 to solve expected consumption maximum utility functions by Merton. The second one is duality method suggested by Xu and Shreve[18] in 1992; compared with conjecture-verification method, the duality method lies its biggest advantage in that this method can directly solve many types of utility functions, such as CARA utility functions, and power utility functions.

2. Uutility Indifference Pricing Model for Highway Government Guarantee Option

This section will build utility indifference pricing model for the European option of minimum traffic government guarantee, with the comprehensive consideration of expected consumption utility and ultimate wealth utility maximization principle.

Suppose there are three assets, in which two are risk assets (tradable equity portfolio and non-tradable underlying assets (highway toll)), and the other one is risk-free assets (interest rate is r).

Suppose the price of tradable risk assets (stock) complies with the following Geometric Brownian Motion(GBM):

$$\frac{dP_s}{P_s} = \mu^P ds + \sigma^P dB_s^P \qquad t \le s, \quad P_t = p > 0 \tag{0.1}$$

And non-tradable underlying assets (traffic flow) complies with the following diffusion process:

$$dX_s = \mu_s^X\left(s, X_s\right)ds + \sigma_s^X\left(s, X_s\right)dB_s^X \qquad t \le s, \quad X_t = x > 0 \tag{0.2}$$

μ^P and σ^P are constant, B_s^P and B_s^X are standard independent Brownian Motion.

When the initial wealth is w, risk-free assets, risk assets and consumption will be allocated by the investors, wealth invested in risk-free assets is $\pi_s^r\left(t \le s \le T\right)$, and risky assets is $\pi_s^p\left(t \le s \le T\right)$, so the total wealth should be $W_s = \pi_s^r + \pi_s^p$, and at any time the magnitude meets the following change:

$$dW_s = \left[\pi_s^p \mu^p + \pi_s^r r\right]ds + \pi_s^p \sigma^p dB_s^p - C_s ds = \left[W_s r + \pi_s^p\left(\mu^p - r\right) - C_s\right]ds + \pi_s^p \sigma^p dB_s^p \tag{0.3}$$

In which C_s is consumption in unit time.

When the investor doesn't invest in highway government guarantee option, optimal investment and consumption strategy should be determined to maximize the following utility functions:

$$\begin{aligned} &V^0\left(t, w\right) = \sup_{C_s, S_s^P} E\left[\int_t^T e^{-\delta(s-t)} U\left(C_s\right)ds + e^{-\delta(T-t)} U\left(W_T\right) | W_t = w\right] \\ &s.t. \quad dW_s = \left[W_s r + \pi_s^p\left(\mu^p - r\right) - C_s\right]ds + \pi_s^p \sigma^p dB_s^p \end{aligned} \tag{0.4}$$

When the investor purchases one unit of highway minimum traffic government guarantee European option at the initial time, expire date of option is T, minimum guaranteed traffic is K, charge for every vehicle is D, so the maturity value is $H = D\left(X_T - K\right)^+$. Optimal investment and consumption strategy should be determined by the investor to maximize the following utility functions:

$$\begin{aligned} &V(t, w, x) = \sup_{C_s, S_s^P} E\left[\int_t^T e^{-\delta(s-t)} U\left(C_s\right)ds + e^{-\delta(T-t)} U\left(W_T + H\right) | W_t = w, X_t = x\right] \\ &s.t. \quad \begin{cases} dW_s = \left[W_s r + \pi_s^p\left(\mu^p - r\right) - C_s\right]ds + \pi_s^p \sigma^p dB_s^p \\ \qquad H = D\left(X_T - K\right)^+ \end{cases} \end{aligned} \tag{0.5}$$

According to the utility indifference pricing principle, the price $h(x,t)$ of guarantee option should have equal maximum effect whether there is investment to the option or not.

$$V(t,w,x)=V^0\left(t,w+h(x,t)\right) \tag{0.6}$$

3. Utility Indifference Price of Highway Government Guarantee European Option

This section will solve the utility indifference price $h(t,x)$ of this European option with the HJB equation to the highway government guarantee model built in Section 2.

(1) For optimal functions $V^0(t,w)$, its Hamilton-Jacobi-Bellman equations (HJB) should be:

$$V_t^0-\delta V^0+\sup\left\{\left[wr+\pi_t^p\left(\mu^p-r\right)-C_t\right]V_w^0+\frac{1}{2}\left(\pi_t^p\sigma^p\right)^2V_{ww}^0+U\left(C_t\right)\right\}=0 \tag{1.1}$$

In which optimal investment and consumption strategy should meet a first order condition:

$$U_1'=V_w\Rightarrow C_t^*=I(V_w^0)\quad (I\ \text{ is inverse functions of } U') \tag{1.2}$$

$$\left(\mu^p-r\right)V_w^0+\left(\pi_t^p\right)^*\left(\sigma^p\right)^2V_{ww}^0=0\Rightarrow\left(\pi_t^p\right)^*=-\frac{\left(\mu^p-r\right)V_w^0}{\left(\sigma^p\right)^2V_{ww}^0} \tag{1.3}$$

Substitute Equ.(1.2)(1.3) in Equ.(1.1) to get

$$V_t^0-\delta V^0+\left[W_tr-\frac{\left(\mu^p-r\right)^2V_w^0}{\left(\sigma^p\right)^2V_{ww}^0}-I(V_w^0)\right]V_w^0+\frac{1}{2}\frac{\left(\mu^p-r\right)^2\left(V_w^0\right)^2}{\left(\sigma^p\right)^2V_{ww}^0}+U\left[I\left(V_w^0\right)\right]=0 \tag{1.4}$$

After simplification the Equ.(1.4) becomes:

$$V_t^0-\delta V^0+\left[W_tr-I(V_w^0)\right]V_w^0-\frac{1}{2}\frac{\left(\mu^p-r\right)^2\left(V_w^0\right)^2}{\left(\sigma^p\right)^2V_{ww}^0}+U\left[I\left(V_w^0\right)\right]=0 \tag{1.5}$$

(2) For optimal functions $V(t,w,x)$, its Hamilton-Jacobi-Bellman equations (HJB) should be:

$$\begin{aligned}&V_t-\delta V+\sup\left\{\left[wr+\pi_t^p\left(\mu^p-r\right)-C_t\right]V_w+\frac{1}{2}\left(\pi_t^p\sigma^p\right)^2V_{ww}+U\left(C_t\right)\right\}\\&+\mu_t^XV_x+\frac{1}{2}V_{xx}\left(\sigma_t^X\right)^2=0\end{aligned} \tag{1.6}$$

In which optimal investment and consumption strategy should meet a first order condition:

$$U_1'=V_w\Rightarrow C_t^*=I(V_w) \tag{1.7}$$

$$\left(\mu^p-r\right)V_w+\left(\pi_t^p\right)^*\left(\sigma^p\right)^2V_{ww}=0\Rightarrow\left(\pi_t^p\right)^*=-\frac{\left(\mu^p-r\right)V_w}{V_{ww}} \tag{1.8}$$

Substitute Equ.(1.7) (1.8)in Equ.(1.6) to get:

$$V_t-\delta V+\left[W_tr-I(V_w)\right]V_w-\frac{1}{2}\frac{\left(\mu^p-r\right)^2\left(V_w\right)^2}{V_{ww}}+U\left[I\left(V_w\right)\right]+\mu_t^XV_x+\frac{1}{2}V_{xx}\left(\sigma_t^X\right)^2=0 \tag{1.9}$$

(3) According to the utility indifference pricing principle $V(w,x,t)=V^0(w+h(x,t),t)$, the partial derivative should satisfy:

$V_t=V_t^0+V_w^0 h_t$, $V_w=V_w^0$, $V_{ww}=V_{ww}^0$, $V_x=V_w^0 h_x$, $V_{xx}=V_{ww}^0 h_x^2+V_w^0 h_{xx}$

The above equations are substituted in Equ.(1.9) with the result of:

$$V_t^0+V_w^0 h_t-\delta V+\left[wr-I(V_w^0)\right]V_w^0-\frac{1}{2}\frac{\left(\mu^p-r\right)^2\left(V_w^0\right)^2}{V_{ww}^0}+U\left[I\left(V_w^0\right)\right]+\mu_t^X V_w^0 h_x$$
$$+\frac{1}{2}\left(V_{ww}^0 h_x^2+V_w^0 h_{xx}\right)\left(\sigma_t^X\right)^2=0 \tag{1.10}$$

Replace w with $w+h$ in Equ.(1.5):

$$V_t^0-\delta V^0+\left[\left(w+h\right)r-I(V_w^0)\right]V_w^0-\frac{1}{2}\frac{\left(\mu^p-r\right)^2\left(V_w^0\right)^2}{\left(\sigma^p\right)^2 V_{ww}^0}+U\left[I\left(V_w^0\right)\right]=0 \tag{1.11}$$

From Equ.(1.10)(1.11) there is:

$$V_w^0 h_t-hrV_w^0+\mu_t^X V_w^0+\frac{1}{2}\left(V_{ww}^0 h_X{}^2+V_w^0 h_{xx}\right)\left(\sigma_t^X\right)^2=0 \tag{1.12}$$

After simplification, the utility indifference price $h(t,x)$ of highway government guarantee European option satisfies following partial differential equation:

$$rh=h_t+\mu_t^X h_x+\frac{1}{2}\frac{V_{ww}^0}{V_w^0}\left(\sigma_t^X\right)^2 h_x{}^2+\frac{1}{2}\left(\sigma_t^X\right)^2 h_{xx} \tag{1.13}$$

The boundary condition is:

$$h(T,X)=H \tag{1.14}$$

4. Model Solution under CARA Utility and Analysis

This study inherits the advantages of utility indifference pricing, and is available to solve utility indifference prices under kinds of utility functions according to the result of last section (such as CARA utility functions, power utility functions). For the next section, the utility indifference price of highway government guarantee European option will be analyzed under the condition of the utility functions being constant absolute risk aversion (CARA), and so will the relation between the price and investors' risk aversion coefficient.

4.1 Model Solution for CARA Utility

Suppose the investor meets the following CARA functions:

$$U\left(C_t\right)=-\frac{e^{-\alpha C_t}}{\alpha},U\left(W_t\right)=-\frac{e^{-\alpha W_t}}{\alpha}$$

In which α is absolute risk aversion coefficient, bigger the value is, stronger risk aversion awareness of the decision maker will be.

With duality method (Xu and Shreve 1992)[18] the following equation can be got (please to the attachment for details)

$$V^0(t,w)=e^{\frac{-\alpha r}{1+(r-1)e^{-r(T-t)}}w-\frac{\alpha r f(t)}{1+(r-1)e^{-r(T-t)}}}\left[\frac{e^{-r(T-t)}}{r\alpha}-\frac{1}{r\alpha}-\frac{1}{\alpha}e^{-r(T-t)}\right] \quad (2.1)$$

It can be easily got from Equ.(2.1) that：

$$\frac{V_{ww}^0}{V_w^0}=-\frac{\alpha r}{1+(r-1)e^{-r(T-t)}} \quad (2.2)$$

Equ.(2.1)(2.2) is substituted in Equ.(1.13)(1.14) with the result of partial differential equation satisfied by the utility indifference price of highway government guarantee European option under CARA utility：

$$rh=h_t+\mu_t^X h_x-\frac{1}{2}\frac{\alpha r}{1+(r-1)e^{-r(T-t)}}h_x^{\ 2}+\frac{1}{2}\left(\sigma_t^X\right)^2 h_{xx} \quad (2.3)$$

The boundary condition is：

$$h(T,X_T)=H \quad (2.4)$$

4.2 Analysis of Utility Indifference Price

(1) When risk aversion coefficient $\alpha=0$

From Equ. (2.3) there is：

$$rh=h_t+\mu_t^X h_x+\frac{1}{2}\left(\sigma_t^X\right)^2 h_{xx} \quad (2.5)$$

At present, it's the same with traditional option pricing B-S equation under the risk neutral condition, which means the highway government guarantee European option pricing method in this research is suitable for both complete and incomplete market. There will be no more analysis to this situation.

(2) When risk aversion coefficient $\alpha\neq 0$

The partial differential equation in this research satisfied by the utility indifference price is just a general form, suppose X_s complies with Geometric Brownian Motion, that is $\mu_s^X=X_s\mu^X$, $\sigma_s^X=X_s\sigma^X$, from Equ.(2.3) there is:

$$rh=h_t+x\mu^X h_x-\frac{1}{2}\frac{\alpha r}{1+(r-1)e^{-r(T-t)}}h_x^{\ 2}+\frac{1}{2}x^2\left(\sigma^X\right)^2 h_{xx} \quad (2.6)$$

Finite difference method is used to solve the partial differential equation Equ.(2.6) under the boundary condition of Equ.(2.4). Fig.1 shows, in different risk aversion coefficient, the variation law with time t for the highway government guarantee minimum flow European option price. As can be seen from the graph:

1) The utility indifference price of guarantee option decreases when risk aversion coefficient increases, it decreases from 7.45 when $\alpha=0$ (same with B-S equation) to 2.66 when $\alpha=1$, and down to 1.24 when $\alpha=10$. This is because the utility indifference method has been introduced to this research for pricing of options, which has broken the risk neutral assumption in traditional B-S pricing. When the investors feel aversion to the risk, the utility indifference price of options will be reduced naturally, and the bigger the risk aversion coefficient α is, the greater the decreased degree will be.

2) When risk aversion coefficient is small($\alpha \le 4$), and other things being equal, the farther the present moment t is away from settlement moment T, the higher the utility indifference price of option will be, which is the same result of traditional B-S equation. Because with the extension of settlement moment T, the uncertainty of payment at this moment will be increased, so will be the possibility for investors to get higher return, therefore the price of option increases.

3) When risk aversion coefficient is big($\alpha \ge 7$), and other things being equal, the farther the present moment t is away from settlement moment T, the lower the utility indifference price of option will be, which is total opposite to the result of traditional B-S equation. Because with the extension of settlement moment T, the uncertainty of income will raise the price of option, but as the investors hold a high risk aversion attitude, they would like to avoid the risk and lower the option price. The price raised by income uncertainty can't balance the price lowered by risk aversion, so in general the option price will be lowered.

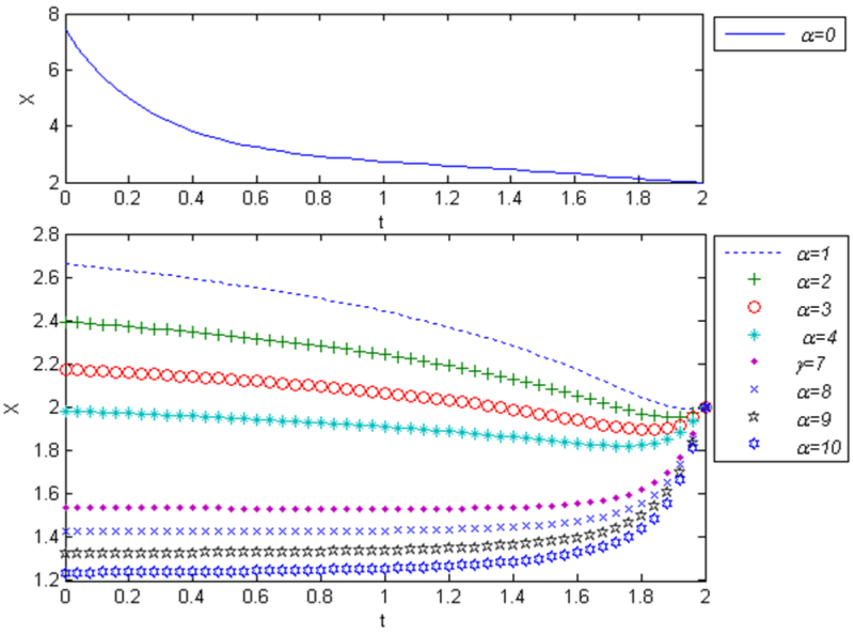

Fig. 1. The variation law with time difference for the highway government guarantee minimum flow European option price in different risk aversion coefficient

Parameter setting: suppose the toll for every vehicle $D = 1$, the expiration time of option $T = 2$, the underlying price of minimum traffic guarantee option (traffic flow) $x = 8$, and the exercise price of minimum traffic guarantee option (traffic flow) $K = 6$, $\mu^X = 0.05$, $\sigma^X = 0.5$, $r = 0.05$.

Readers with interest can similarly analyze the relation between the utility indifference price of guarantee option and risk aversion coefficient when X_s complies with mean reversion and CIR process, which will be not discussed here anymore.

5. Conclusion

This research has broken the traditional risk neutral assumption and introduced the utility indifference pricing to the option pricing of highway government minimum traffic guarantee by considering the investors' risk attitude in incomplete market. The contribution of this paper can be concluded into three points:

1) Takes the idea of comprehensive consideration of expected consumption utility and ultimate wealth utility into consideration, and applies the utility indifference pricing principle to the option pricing of highway government minimum traffic guarantee, finally builds the pricing model of highway minimum traffic guarantee European option.

2) Offers the partial differential equation for utility indifference price by solving the pricing model with HJB equation and duality method, this partial differential equation is available for option pricing under different utility functions. By analyzing the utility indifference price partial differential equation under CARA utility functions, the pricing method offered in this research has been proved to be same with traditional B-S equation under the condition of investors being risk neutral. To sum up, the pricing method in this paper has got the advantage of utility indifference pricing, which is the pricing method is not only suitable for incomplete market, but also for complete market.

3) By analyzing the utility indifference price under CARA utility functions with finite difference method, an interesting phenomenon has been found. Different from the monotone increasing of European option price under the increase of option duration in complete market, in incomplete market, there is negative correlation between investors' risk aversion attitude and utility indifference price of government guarantee, and the negative effect will expand with the growth of investors' risk aversion coefficient, when the risk aversion coefficient grows to a certain stage, the utility indifference price might even decrease along with the increase of option duration.

References

[1] Klein, M., Managing guarantee programs in support of infrastructure investment. World Bank Policy Research Working Paper No. 1812, 1997.

[2] Llanto, G.M., M.C. Soriano and G. Cecilia, Government guarantees in infrastructure projects: a second, third look at the policy. Philippine Institute for Development Studies Policy Notes, 1997(97-11).

[3] Llanto, G., Managing Government Guarantees and Contingent Liabilities. Philippine Institute for Development Studies, 2000(2000-17).

[4] Myers, S.C., Determinants of corporate borrowing. Journal of financial economics, 1977. 5(2), 147-175.

[5] Wibowo, A., Valuing guarantees in a BOT infrastructure project. Engineering, Construction and Architectural Management, 2004. 11(6), 395-403.

[6] Huang, H., C. Chern and C. Lin, EFL learners' online reading strategies: A comparison between high and low EFL proficient readers. 2006.

[7] Brandao, L.E.T. and E. Saraiva, The option value of government guarantees in infrastructure projects. Construction Management and Economics, 2008. 26(11), 1171-1180.

[8] Galera, A. and A.S. Solino, A Real Options Approach for the Valuation of Highway Concessions. Transportation Science, 2010. 44(3), 416-427.

[9] Zhang, G.X., et al., Non-competition Guaranteed value and effect on the Toll Road Project. Transport, 2011. 26(2), 178-184.

[10] Dixit, A.K. and R.S. Pindyck, Investment under uncertainty: Princeton University Press, 1994.

[11] Trigeorgis, L., Real options in capital investment: Models, strategies, and applications: Praeger Publishers, 1995.

[12] Hodges, S.D. and A. Neuberger, Optimal replication of contingent claims under transaction costs. Review of futures markets, 1989. 8(2), 222-239.

[13] Davis, M.H.A., Option pricing in incomplete markets. 1997, Cambridge Univ. Press. p. 216-226.

[14] Musiela, M. and T. Zariphopoulou, An example of indifference prices under exponential preferences. Finance and Stochastics, 2004. 8(2), 229-239.

[15] Henderson, V., Valuing the option to invest in an incomplete market. Mathematics and Financial Economics, 2007. 1(2), 103-128.

[16] Ewald, C. and Z. Yang, Utility based pricing and exercising of real options under geometric mean reversion and risk aversion toward idiosyncratic risk. Mathematical Methods of Operations Research, 2008. 68(1), 97.

[17] Miao, J.J. and N. Wang, Investment, consumption, and hedging under incomplete markets. Journal of Financial Economics, 2007. 86(3), 608-642.

[18] Xu, G.L. and S.E. Shreve, A duality method for optimal consumption and investment under short-selling prohibition. II. Constant market coefficients. The Annals of Applied Probability, 1992. 2(2), 314-328.

Appendix

Solve $V^0(t,w)$ with duality method (Xu and Shreve 1992) under the condition of CARA

Xu&Shreve（1992）has proved that $\tilde{V}^0(t,y)$, the dual functions of $V^0(t,w)$ meets：

$$\tilde{V}^0(t,y) = G(t,y) - S(t,y) \qquad \text{(A.7)}$$

$$G(t,y) \overset{\Delta}{=} E\left(\int_t^T e^{-\delta(s-t)} U\left[I\left(y\zeta(t,s)\right)\right] ds + e^{-\delta(T-t)} U\left[I\left(y\zeta(t,T)\right)\right]\right) \qquad \text{(A.8)}$$

$$S(t,y) \overset{\Delta}{=} E\left(\int_t^T e^{-\delta(s-t)} y\zeta(t,s) I\left(y\zeta(t,s)\right) ds + e^{-\delta(T-t)} y\zeta(t,T) I\left(y\zeta(t,T)\right)\right) \qquad \text{(A.9)}$$

In which $U(C_s) = -\frac{e^{-\alpha C_s}}{\alpha}$, $U'(C_s) = e^{-\alpha C_s}$, inverse functions of U' is $I(y) = -\frac{\ln y}{\alpha}$

$\zeta(t,s) \overset{\Delta}{=} e^{(\delta - r)(s-t)} Z(t,s)$, $Z(t,s) \overset{\Delta}{=} e^{-\theta\left[B_s^P - B_t^P\right] - \frac{1}{2}\theta^2 (s-t)}$, $\theta = (\sigma)^{-1}\left(\mu^P - r\right)$

$$U\left[I\left(y\zeta\right)\right] = -\frac{y\zeta}{\alpha}$$

(1) The above equations are substituted in Equ.(A.8) to get $G(t,y)$：

$$G(t,y)\overset{\Delta}{=}E\left(\int_t^T -e^{-\delta(s-t)}\frac{y\zeta(t,s)}{\alpha}ds-e^{-\delta(T-t)}\frac{y\zeta(t,T)}{\alpha}\right)$$
$$=\int_t^T -e^{-\delta(s-t)}\frac{yE(\zeta(t,s))}{\alpha}ds-e^{-\delta(T-t)}\frac{yE(\zeta(t,T))}{\alpha}$$
$$=\int_t^T -e^{-\delta(s-t)}\frac{y}{\alpha}e^{(\delta-r)(s-t)}ds-e^{-\delta(T-t)}\frac{y}{\alpha}e^{(\delta-r)(s-t)}$$
$$=\frac{e^{-r(T-t)}}{r\alpha}y-\frac{y}{r\alpha}-\frac{y}{\alpha}e^{-r(T-t)} \tag{A.10}$$

(2) Substitute the above equations in Equ.(A.9) to get $S(t,y)$:

$$S(t,y)\overset{\Delta}{=}=E\left(\int_t^T e^{-\delta(s-t)}y\zeta(t,s)\frac{-\ln y\zeta(t,s)}{\alpha}ds+e^{-\delta(T-t)}y\zeta(t,T)\frac{-\ln y\zeta(t,T)}{\alpha}\right)$$
$$=\int_t^T \text{-}\frac{y}{\alpha}e^{-\delta(s-t)}E[\zeta(t,s)\ln y\zeta(t,s)]ds\text{-}\frac{y}{\alpha}e^{-\delta(T-t)}E[\zeta(t,T)\ln y\zeta(t,T)] \tag{A.11}$$

(2)-1 In order to get Equ.(A.11), $E[\zeta(t,s)\ln\zeta(t,s)]$ should be solved first

$$\zeta(t,s)\ln\zeta(t,s)=e^{(\delta-r)(s-t)}Z(t,s)[(\delta-r)(s-t)+\ln Z(t,s)]$$
$$=(\delta-r)(s-t)e^{(\delta-r)(s-t)}Z(t,s)+e^{(\delta-r)(s-t)}Z\ln Z(t,s)$$

$dZ(t,s)=-Z(t,s)\theta dB_s^P$, $d\ln Z(t,s)=-\theta dB_s^P-\frac{1}{2}\theta^2 ds$

$$d(Z(t,s)\ln Z(t,s))=\ln Z(t,s)dZ(t,s)-\theta Z(t,s)dB_s^P+\frac{1}{2}\theta^2 Z(t,s)ds$$

From all above equations there has:

$$E(\zeta(t,s)\ln\zeta(t,s))=(\delta-r)(s-t)e^{(\delta-r)(s-t)}+e^{(\delta-r)(s-t)}\frac{1}{2}\theta^2(s-t)$$
$$=e^{(\delta-r)(s-t)}(s-t)\left(\delta-r+\frac{1}{2}\theta^2\right) \tag{A.12}$$

(2)-2 Substitute Equ.(A.12) in Equ.(A.11) to get

$$S(t,y)=\int_t^T -\frac{y}{\alpha}e^{-\delta(s-t)}\left[e^{(\delta-r)(s-t)}\ln y+e^{(\delta-r)(s-t)}(s-t)\left(\delta-r+\frac{1}{2}\theta^2\right)\right]ds$$
$$-\frac{y}{\alpha}e^{-\delta(T-t)}\left[e^{(\delta-r)(T-t)}\ln y+e^{(\delta-r)(T-t)}(T-t)\left(\sigma-r+\frac{1}{2}\theta^2\right)\right]$$
$$=\left(-\frac{y}{\alpha}\right)\left[\frac{1-e^{-r(T-t)}}{r}\ln y+\frac{1-[1+r(T-t)]e^{-r(T-t)}}{r^2}\left(\delta-r+\frac{1}{2}\theta^2\right)\right]$$
$$-\frac{y}{\alpha}\left[e^{-r(T-t)}\ln y+e^{-r(T-t)}(T-t)\left(\delta-r+\frac{1}{2}\theta^2\right)\right] \tag{A.13}$$

(3) Substitute Equ.(A.10)(A.13) in Equ.(A.7) to get

$$\tilde{V}_y^0=G_y-S_y=\frac{1}{\alpha}\left[\frac{1+(r-1)e^{-r(T-t)}}{r}\right]\ln y+f(t) \tag{A.14}$$

$$f(t)\overset{\Delta}{=}\frac{1}{\alpha}\left[\frac{1-\left[1+r(T-t)\right]e^{-r(T-t)}}{r^2}+e^{-r(T-t)}\left(T-t\right)\right]\left(\delta-r+\frac{1}{2}\theta^2\right) \quad \text{(A.15)}$$

Xu&Shreve（1992）has proved

$$V^0\left(t,w\right)=G\left(t,Y\left(t,w\right)\right) \quad \text{(A.16)}$$

In which $Y\left(t,w\right)$ is inverse functions of $-\tilde{V}_y^0(t,\cdot)$

$$Y\left(t,w\right)=e^{\frac{-\alpha r}{1+(r-1)e^{-r(T-t)}}w-\frac{\alpha r f(t)}{1+(r-1)e^{-r(T-t)}}} \quad \text{(A.17)}$$

Substitute Equ.(A.17) in Equ.(A.16) to get：

$$V^0\left(t,w\right)=e^{\frac{-\alpha r}{1+(r-1)e^{-r(T-t)}}w-\frac{\alpha r f(t)}{1+(r-1)e^{-r(T-t)}}}\left[\frac{e^{-r(T-t)}}{r\alpha}-\frac{1}{r\alpha}-\frac{1}{\alpha}e^{-r(T-t)}\right] \quad \text{(A.18)}$$